ISSN 2348 – 8034
Impact Factor- 3.155

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

A STUDY OF PREY-PREDATOR SYSTEM WITH VERTICAL TRANSMITTED DISEASE IN PREDATOR

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ABSTRACT

The present paper dials with the proposed and analyzed of an eco-epidemiological system consisting of prey-predator system involving infectious disease in predator population that transmitted vertically within predator population only. Nonlinear functional response is used to describe the predation process. All possible equilibrium points are determined. The local stability analysis of these points are studied. The basin of attraction of each point is also investigated. It is observed that the existence of vertically transmitted disease in predator caused to extinction to predator population and then the system losses their persistence.

Keywords: Prey-predator, vertical transmitted disease, stability, basin of attraction.

I. INTRODUCTION

It is well known that diseases influence both the dynamics of their hosts as well as the dynamics of interacting species like predators, prey and competitors. Further, interacting species can influence disease dynamics by altering the host's dynamics. Therefore prey-predator interactions and disease in the system affects each other [1-7]. The combination of these two effects is often called eco-epidemiology, the interaction of ecology and epidemiology.

Most of the previous studies of infectious diseases were observed that the disease in animal populations cussed that mortality of the host population or reduced reproduction of the host populations in their natural habitats [6-9]. Indeed reduced population sizes and destabilization of equilibrium into oscillations are caused by the presence of infectious disease in one or both of the populations.

Many of the eco-epidemiological studies are restricted to the situations where the infectious disease presence in the prey species only [6, 9-11]. Few investigations take in to account the spread of disease from prey to predator through predation process of infected prey [12-15]. On the other hand there are some investigations about prey-predator model with disease in the predator population. Haque [16] proposed a prey-predator model includes SIS parasitic infection in the predator population with linear functional response and nonlinear disease incidence rate. Haque and Venturino [17] considered a prey-predator model with SI epidemic disease spread in predators involving linear functional response. Das [18] studied a prey-predator model with SI epidemic disease in predators included Holling type-II as a functional response. Venturino [6] proposed and analyzed prey-predator model with SIS disease in predators included linear functional response and linear disease incidence. Haque and Venturino [19] considered a prey-predator model with SI epidemic disease spread in predators included ratio-dependent functional response and linear disease's incidence rate. Recently, Naji and Yaseen [20] proposed and analyzed a mathematical model describing prey-predator model having SIS epidemic disease in the predator population with nonlinear functional response, represented by Holling type-II and ratio-dependent incidence rate.

Keeping the above in view, in this research work, an eco-epidemiological system consisting of prey-predator model involving SI-type of vertically transmitted disease in predator has been proposed and analyzed. It is assumed that the predator consumed the prey species according to nonlinear functional response that given by Cosner et al. [21]. The structure of the present paper is arranged as follows, the model formulated in next section. Section 3 contains the local stability analysis, while section 4 determine the basin of attraction of each equilibrium point. Finally the discussions and conclusions are given in the section 5.

II. THE MODEL FORMULATION

In this section a prey-predator system with vertically transmitted infectious disease in predator population is mathematically formulated. Now, in order to formulates the model that describe the dynamics of such real world system the following hypotheses are considered

1. Let $X(t)$ denotes the density of the prey species at time t, $Y(t)$ is the population density of the susceptible predator at time t and $Z(t)$ represents the population density of the infected predator at time t.

2. It is assumed that in the absence of the predator the prey species grows logistically with intrinsic growth rate $r > 0$ and carrying capacity $K > 0$.
3. The predators populations, represented by $Y(t)$ and $Z(t)$, consume the prey according to the predator dependent type of functional response, that originally proposed by Cosner et al. [21], in which $a > 0$ and $c > 0$ denote the attack rates respectively while $b > 0$ represents the predator half saturation constant.
4. It is assumed that the disease in predator population, is transferred from parent to offspring, which is known vertical transmitted. Furthermore, the disease is also transmitted from infected predator to susceptible predator by contact according to mass action law with infected rate $\lambda > 0$.
5. The predators decay exponentially in the absence of the prey species with natural death rate $\gamma_1 > 0$, while the disease causes extra death within the population for infected predator represented by $\gamma_2 > 0$.

Accordingly the dynamics of the prey-predator system with infectious disease in predator that described above can be represented mathematically by the following set of nonlinear ordinary differential equations:

$$\begin{aligned}\frac{dX}{dt} &= rX\left(1-\frac{X}{K}\right)-\frac{aXY^2}{b+XY}-\frac{cXZ^2}{b+XZ}\\ \frac{dY}{dt} &= e_1\frac{aXY^2}{b+XY}-\lambda YZ-\gamma_1 Y\\ \frac{dZ}{dt} &= e_2\frac{cXZ^2}{b+XZ}+\lambda YZ-(\gamma_1+\gamma_2)Z\end{aligned} \tag{1}$$

with $X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0$ and $0 < e_i < 1$ for $i = 1,2$ denote to the conversion rates of Y and Z respectively. Recall that, due to the biological meaning of the variables given in system (1) the system defines on the following domain $R_+^3 = \{(X, Y, Z) \in R^3 : X \geq 0, Y \geq 0, Z \geq 0\}$.

Note that the interaction functions in the right hand side of system (1) are continuous and have continuous partial derivatives, and hence they are Liptchazian functions. Therefore system (1) has a unique solution. Moreover the all solutions of system (1) are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1) that initiate in R_+^3 are uniformly bounded.

Proof: Let $(X(t), Y(t), Z(t))$ be any solution initiate in R_+^3. Since we have that

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right)$$

Then straightforward computation shows that $X \leq K$ as $t \to \infty$. Now consider the function $M = X + Y + Z$, then we obtain that

$$\frac{dM}{dt} + \rho M \leq 2r$$

Here $\rho = min.\{r, \gamma_1\}$. So, straightforward computation gives that $M \leq \frac{2r}{\rho}$ as $t \to \infty$. Hence all solutions are uniformly bounded. ■

III. LOCAL STABILITY ANALYSIS

In this section the existence conditions of all possible equilibrium points of system (1) are established and then their local stability analyses are discussed. There are at most five non-negative equilibrium points of system (1), these are described as follows:

The vanishing equilibrium point that denoted by $E_0 = (0,0,0)$ and the predator free equilibrium point, say $E_1 = (K, 0,0)$, on the X −axis, are always exist.

The disease free equilibrium point $E_2 = (\bar{X}, \bar{Y}, 0)$ where

$$\bar{X} = \frac{b\gamma_1}{(ae_1 - \gamma_1)\bar{Y}} \quad (2)$$

while $\bar{Y}$ is a positive root of the third degree polynomial that given by

$$Y^3 - \frac{rbe_1}{ae_1 - \gamma_1}Y + \frac{rb^2 e_1 \gamma_1}{K(ae_1 - \gamma_1)^2} = 0 \quad (3)$$

Now using the method of Cardano for finding the root, its easy to verify that, Eq. (3) has a unique positive root provided that

$$27b\gamma_1{}^2 > 4K^2 re_1(ae_1 - \gamma_1) \quad (4a)$$

This root take the form $\left(\alpha_1 - \frac{\beta_1}{\alpha_1}\right)$, where α_1 denotes one of the three roots of the equation

$$\frac{1}{2}\left[-\mu_1 + \sqrt{\mu_1{}^2 + 4\beta_1}\right]^{\frac{1}{3}} = 0 \quad (4b)$$

here $\mu_1 = \frac{rb^2 e_1 \gamma_1}{K(ae_1 - \gamma_1)^2}$; $\beta_1 = -\frac{rbe_1}{3(ae_1 - \gamma_1)}$.

Moreover, for positivity of $\bar{X}$ we should have

$$ae_1 > \gamma_1 \quad (4c)$$

Thus its clear that conditions (4a) and (4c) are the necessary and sufficient conditions for the existence of E_2.

Similarly, the susceptible predator free equilibrium point $E_3 = (\hat{X}, 0, \hat{Z})$ where

$$\hat{X} = \frac{b(\gamma_1 + \gamma_2)}{(ce_2 - (\gamma_1 + \gamma_2))\hat{Z}} \quad (5)$$

while $\hat{Z}$ is a positive root of the third degree polynomial that given by

$$Z^3 - \frac{rbe_2}{ac - (\gamma_1 + \gamma_2)}Z + \frac{rb^2 e_2(\gamma_1 + \gamma_2)}{K(ce_2 - (\gamma_1 + \gamma_2))^2} = 0 \quad (6)$$

Now again by using the method of Cardano for finding the root, it's easy to verify that, Eq. (6) has a unique positive root provided that

$$27b(\gamma_1 + \gamma_2)^2 > 4K^2 re_2(ce_2 - (\gamma_1 + \gamma_2)) \quad (7a)$$

This root take the form $\left(\alpha_2 - \frac{\beta_2}{\alpha_2}\right)$, where α_2 denotes one of the three roots of the equation

$$\frac{1}{2}\left[-\mu_2 + \sqrt{\mu_2{}^2 + 4\beta_2}\right]^{\frac{1}{3}} = 0 \quad (7b)$$

here $\mu_2 = \frac{rb^2 e_2(\gamma_1 + \gamma_2)}{K(ce_2 - (\gamma_1 + \gamma_2))^2}$; $\beta_2 = -\frac{rbe_2}{3(ce_2 - (\gamma_1 + \gamma_2))}$.

Moreover, for positivity of $\hat{X}$ we should have

$$ce_2 > \gamma_1 + \gamma_2 \quad (7c)$$

ISSN 2348 – 8034
Impact Factor- 3.155

Thus it's clear that conditions (7a) and (7c) are the necessary and sufficient conditions for the existence of E_3.

On other hand there is no equilibrium point in the YZ −plane, due to the fact that the predator can't survive without its food.

Finally the positive equilibrium point $E_4 = (X^*, Y^*, Z^*)$ exists uniquely in the interior of R^3_+ provided that there is a positive solution to the following algebraic system of equations.

$$\begin{aligned} f_1(X,Y,Z) &= r\left(1-\frac{X}{K}\right) - \frac{aY^2}{b+XY} - \frac{cZ^2}{b+XZ} = 0 \\ f_2(X,Y,Z) &= e_1\frac{aXY}{b+XY} - \lambda Z - \gamma_1 = 0 \\ f_3(X,Y,Z) &= e_2\frac{cXZ}{b+XZ} + \lambda Y - (\gamma_1+\gamma_2) = 0 \end{aligned} \tag{8a}$$

Straightforward computation shows that system (8) has a unique positive solution given by

$$Z^* = \frac{1}{\lambda}\left[\frac{ae_1X^*Y^*}{b+X^*Y^*} - \gamma_1\right] \tag{8b}$$

here (X^*, Y^*) represents a positive intersection point between the following two isoclines:

$$\begin{aligned} f(X,Y) &= M_1X^2Y^2 + M_2X^2Y + M_3XY^2 + M_4XY + M_5X + M_6Y + M_7 = 0 \\ g(X,Y) &= N_1X^4Y^2 + N_2X^3Y^2 + N_3X^3Y + N_4X^2Y^3 \\ &\quad + N_5X^2Y^2 + N_6X^2Y + N_7X^2 + N_8XY + N_9XY^2 \\ &\quad + N_{10}XY^3 + N_{11}X + N_{12}Y^2 + N_{13} = 0 \end{aligned}$$

where $M_1 = \lambda\gamma_1 - ae_1\lambda$, $M_2 = ace_1e_2 - ce_2\gamma_1 + \gamma_1(\gamma_1+\gamma_2) - ae_1(\gamma_1+\gamma_2)$, $M_3 = -b\lambda^2$, $M_4 = -b\lambda\gamma_2$, $M_5 = \lambda\gamma_1(\gamma_1+\gamma_2-ce_2)$, $M_6 = -b\lambda^2$ and $M_7 = -b\lambda(\gamma_1+\gamma_2)$. While $N_1 = r\lambda(\gamma_1 - e_1a)$, $N_2 = r\lambda[K(e_1a - \gamma_1) - \lambda b]$, $N_3 = r\lambda b(2\gamma_1 - e_1a)$, $N_4 = \lambda Ka(\gamma_1 - e_1a)$, $N_5 = K[r\lambda^2b + 2ce_1a\gamma_1 - c(\gamma_1^2 + e_1^2a^2)]$, $N_6 = r\lambda b[Ke_1a - 2K\gamma_1 - 2\lambda b]$, $N_7 = r\lambda b^2\gamma_1$, $N_8 = 2Kb[r\lambda^2b + c\gamma_1(e_1a - \gamma_1)]$, $N_9 = \lambda Kab\gamma_1$, $N_{10} = -\lambda^2Kab$, $N_{11} = -r\lambda b^2(K\gamma_1 + \lambda b)$, $N_{12} = -\lambda^2Kab^2$ and $N_{13} = Kb^2(r\lambda^2b - \gamma_1^2c)$.

Clearly as $Y \to 0$ then the first isocline $f(X,Y)$ intersect the X −axis at a positive point given by $X_1 = -\frac{M_7}{M_5}$ provided that

$$\gamma_1 + \gamma_2 > ce_2 \tag{8c}$$

However as $Y \to 0$ in the second isocline $g(X,Y)$ we obtain the following second order equation $N_7X^2 + N_{11}X + N_{13} = 0$, which intersects the X −axis at a positive point given by $X_2 = -\frac{N_{11}}{2N_7} + \frac{1}{2N_7}\sqrt{N_{11}^2 - 4N_7N_{13}}$ provided that

$$r\lambda^2 b < \gamma_1^2 c \tag{8d}$$

Therefore it's easy to verify that the above two isoclines have a unique positive intersection point given by (X^*, Y^*) provided that the following conditions are satisfied:

$$\begin{aligned} & X_1 < X_2 \\ & \frac{dY}{dX} = -\frac{\frac{\partial f}{\partial X}}{\frac{\partial f}{\partial Y}} > 0 \\ & \frac{dY}{dX} = -\frac{\frac{\partial g}{\partial X}}{\frac{\partial g}{\partial Y}} < 0 \end{aligned} \tag{8e}$$

Moreover, Z^* will be positive if and only if

$$\frac{ae_1X^*Y^*}{b+X^*Y^*} > \gamma_1 \tag{8f}$$

Accordingly, system (1) has a unique positive equilibrium point that is given by $E_4 = (X^*, Y^*, Z^*)$ if the conditions (8c)-(8f) are satisfied.

Now the general Jacobian matrix of system (1) at the point (X, Y, Z), can be written as:

$$J(X,Y,Z) = \begin{pmatrix} f_1 + X\frac{\partial f_1}{\partial X} & X\frac{\partial f_1}{\partial Y} & X\frac{\partial f_1}{\partial Z} \\ Y\frac{\partial f_2}{\partial X} & f_2 + Y\frac{\partial f_2}{\partial Y} & Y\frac{\partial f_2}{\partial Z} \\ Z\frac{\partial f_3}{\partial X} & Z\frac{\partial f_3}{\partial Y} & f_3 + Z\frac{\partial f_3}{\partial Z} \end{pmatrix} \tag{9}$$

where $\frac{\partial f_1}{\partial X} = -\frac{r}{K} + \frac{aY^3}{(b+XY)^2} + \frac{cZ^3}{(b+XZ)^2}, \frac{\partial f_1}{\partial Y} = -\frac{a(2b+XY)Y}{(b+XY)^2}, \frac{\partial f_1}{\partial Z} = -\frac{c(2b+XZ)Z}{(b+XZ)^2}$

$$\frac{\partial f_2}{\partial X} = \frac{abe_1Y}{(b+XY)^2}, \frac{\partial f_2}{\partial Y} = \frac{abe_1X}{(b+XY)^2}, \frac{\partial f_2}{\partial Z} = -\lambda,$$

$$\frac{\partial f_3}{\partial X} = \frac{cbe_2Z}{(b+XZ)^2}, \frac{\partial f_3}{\partial Y} = \lambda, \frac{\partial f_3}{\partial Z} = \frac{cbe_2X}{(b+XZ)^2}$$

Accordingly, the Jacobian matrix of system (1) at vanishing equilibrium point E_0 is

$$J(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -\gamma_1 & 0 \\ 0 & 0 & -(\gamma_1 + \gamma_2) \end{pmatrix} \tag{10a}$$

Clearly, the eigenvalues of $J(E_0)$ are

$$\sigma_{0X} = r > 0, \sigma_{0Y} = -\gamma_1 < 0, \sigma_{0Z} = -(\gamma_1 + \gamma_2) < 0 \tag{10b}$$

where $\sigma_{0u}; u = X, Y, Z$ represents the eigenvalue of $J(E_0)$ in the u −direction. Therefore the vanishing equilibrium point E_0 is a saddle point.

The Jacobian matrix of system (1) at the predator free equilibrium point E_1 is written

$$J(E_1) = \begin{pmatrix} -r & 0 & 0 \\ 0 & -\gamma_1 & 0 \\ 0 & 0 & -(\gamma_1 + \gamma_2) \end{pmatrix} \tag{11a}$$

Therefore, the eigenvalues of $J(E_1)$ are

$$\sigma_{1X} = -r < 0, \sigma_{1Y} = -\gamma_1 < 0, \sigma_{1Z} = -(\gamma_1 + \gamma_2) < 0 \tag{11b}$$

Consequently, the predator free equilibrium point E_1 is always locally asymptotically stable.

Moreover the Jacobian matrix of system (1) at the disease free equilibrium point $E_2 = (\bar{X}, \bar{Y}, 0)$ can be written as

$$J(E_2) = \begin{pmatrix} \bar{X}\left[-\frac{r}{K} + \frac{a\bar{Y}^3}{\bar{R}_1{}^2}\right] & -\frac{a(2b+\bar{X}\bar{Y})\bar{X}\bar{Y}}{\bar{R}_1{}^2} & 0 \\ \frac{abe_1\bar{Y}^2}{\bar{R}_1{}^2} & \frac{abe_1\bar{X}\bar{Y}}{\bar{R}_1{}^2} & -\lambda\bar{Y} \\ 0 & 0 & \lambda\bar{Y} - (\gamma_1 + \gamma_2) \end{pmatrix} = (b_{ij}) \tag{12a}$$

Thus the eigenvalues of this matrix can be written as:

$$\sigma_{2X}, \sigma_{2Y} = \frac{T_2}{2} \pm \frac{1}{2}\sqrt{T_2 - 4D_2};\ \sigma_{2Z} = \lambda\bar{Y} - (\gamma_1 + \gamma_2) \tag{12b}$$

here $T_2 = \frac{\bar{X}}{K\bar{R}_1{}^2}\left[-r\bar{R}_1{}^2 + aK\bar{Y}(\bar{Y}^2 + be_1)\right]$

$$D_2 = \frac{abe_1\bar{X}\bar{Y}}{K\bar{R}_1{}^4}\left[-r\bar{X}\bar{R}_1{}^2 + 2aK\bar{Y}^2(\bar{X}\bar{Y} + b)\right]$$

with $\bar{R}_1 = b + \bar{X}\bar{Y}$, Hence all the above eigenvalues have negative real parts and then E_2 is locally asymptotically stable if the following conditions hold:

$$\lambda < \frac{\gamma_1 + \gamma_2}{\bar{Y}} \tag{12c}$$

$$aK\,\bar{Y}(\bar{Y}^2 + be_1) < r\bar{R}_1{}^2 < \frac{2aK\bar{Y}^2}{\bar{X}}\bar{R}_1 \tag{12d}$$

On the other hand the Jacobian matrix of system (1) at the susceptible predator free equilibrium point $E_3 = (\hat{X}, 0, \hat{Z})$ can be written as

$$J(E_3) = \begin{pmatrix} \hat{X}\left[-\frac{r}{K} + \frac{c\hat{Z}^3}{\hat{R}_2{}^2}\right] & 0 & -\frac{c(2b+\hat{X}\hat{Z})\hat{X}\hat{Z}}{\hat{R}_2{}^2} \\ 0 & -\lambda\hat{Z} - \gamma_1 & 0 \\ \frac{cbe_2\hat{Z}^2}{\hat{R}_2{}^2} & \lambda\hat{Z} & \frac{cbe_2\hat{X}\hat{Z}}{\hat{R}_2{}^2} \end{pmatrix} = (c_{ij}) \tag{13a}$$

Thus the eigenvalues of this matrix can be written as:

$$\sigma_{3X}, \sigma_{3Z} = \frac{T_3}{2} \pm \frac{1}{2}\sqrt{T_3 - 4D_3};\ \sigma_{3Y} = -\lambda\hat{Z} - \gamma_1 < 0 \tag{13b}$$

where $T_3 = \frac{\hat{X}}{K\hat{R}_2{}^2}\left[-r\hat{R}_2{}^2 + cK\hat{Z}(\bar{Z}^2 + be_2)\right]$

$$D_3 = \frac{cbe_2\hat{X}\hat{Z}}{K\hat{R}_2{}^4}\left[-r\hat{X}\hat{R}_2{}^2 + 2cK\hat{Z}^2\hat{R}_2\right]$$

with $\hat{R}_2 = b + \hat{X}\hat{Z}$, Hence all the above eigenvalues have negative real parts and then E_3 is locally asymptotically stable if the following condition holds:

$$cK\hat{Z}(\hat{Z}^2 + be_2) < r\hat{R}_2{}^2 < \frac{2cK\hat{Z}^2}{\hat{X}}\hat{R}_2 \tag{13c}$$

Finally, The Jacobian matrix of system (1) at the positive equilibrium point $E_4 = (X^*, Y^*, Z^*)$ is given by

$$J(E_4) = (a_{ij}); i, j = 1,2,3 \tag{14a}$$

here $a_{11} = -\frac{rX^*}{K} + \frac{aX^*Y^{*3}}{R_1^{*2}} + \frac{cX^*Z^{*3}}{R_2^{*2}}$; $a_{12} = -\frac{a(2b+X^*Y^*)X^*Y^*}{R_1^{*2}}$, $a_{13} = -\frac{c(2b+X^*Z^*)X^*Z^*}{R_2^{*2}}$; $a_{21} = \frac{e_1abY^{*2}}{R_1^{*2}}$; $a_{22} = \frac{e_1abX^*Y^*}{R_1^{*2}}$; $a_{23} = -\lambda Y^*$; $a_{31} = \frac{e_2bcZ^{*2}}{R_2^{*2}}$; $a_{32} = \lambda Z^*$; and $a_{33} = \frac{e_2bcX^*Z^*}{R_2^{*2}}$ with $R_1^* = b + X^*Y^*$; $R_2^* = b + X^*Z^*$.

Thus the characteristic equation of $J(E_4)$ can be written as:

$$\sigma_4{}^3 + A_1\sigma_4{}^2 + A_2\sigma_4 + A_3 = 0 \tag{14b}$$

with $A_1 = -(a_{11} + a_{22} + a_{33})$,

ISSN 2348 – 8034
Impact Factor- 3.155

$$A_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}$$

$$A_3 = a_{33}(a_{12}a_{21} - a_{11}a_{22}) + a_{23}(a_{11}a_{32} - a_{12}a_{31}) + a_{13}(a_{22}a_{31} - a_{21}a_{32})$$

$$\begin{aligned}\Delta = &-a_{11}a_{22}[a_{11} + a_{22} + 2a_{33}] - (a_{11} + a_{33})[a_{11}a_{33} - a_{13}a_{31}] \\ &+(a_{11} + a_{22})a_{12}a_{21} + (a_{22} + a_{33})[a_{23}a_{32} - a_{22}a_{33}] \\ &+a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\end{aligned}$$

Therefore, by using the Routh-Hurwitz criterion the following theorem, which presents the local stability conditions of E_4, can be proved directly.

Theorem (2): The positive equilibrium point E_4 of system (1) is locally asymptotically stable in R_+^3 provided that the following sufficient conditions are satisfied

$$max.\{\Gamma_1, \Gamma_2\} < \frac{r}{K} < \Gamma_3 \quad (15a)$$

$$\frac{2aY^{*2}}{R_1^*} + \frac{cX^*Z^{*3}}{R_2^{*2}} < \frac{rX^*}{K} < \frac{aX^*Y^{*3}}{R_1^{*2}} + \frac{2c(X^*Z^*+b)Z^{*2}}{R_2^{*2}} \quad (15b)$$

$$e_2bcX^*Z^* < \lambda Y^*R_2^{*2} \quad (15c)$$

$$\Gamma_5 < \Gamma_4 \quad (15d)$$

$$e_1(2b + X^*Z^*) < e_2(2b + X^*Y^*) \quad (15e)$$

Here $\Gamma_1 = \frac{a(Y^{*2}+e_1b)Y^*}{R_1^{*2}} + \frac{c(Z^{*2}+e_2b)Z^*}{R_2^{*2}}$;

$$\Gamma_2 = \frac{aY^{*3}}{R_1^{*2}} + \frac{cZ^{*3}}{R_2^{*2}} + \frac{e_2abc(2b+X^*Y^*)Y^*Z^*}{\lambda R_1^{*2}R_2^{*2}}$$

$$\Gamma_3 = \frac{a\left(Y^{*2}+e_1b\right)Y^*}{R_1^{*2}} + \frac{c\left(Z^{*2}+2e_2b\right)Z^*}{R_2^{*2}};$$

$$\Gamma_4 = \frac{e_1a^2(2b + X^*Y^*)X^*Y^{*2}}{R_1^{*2}}\left[rR_1^{*2}R_2^{*2} - aK\left(Y^{*2} + e_1b\right)Y^*R_2^{*2} - cKZ^{*3}R_1^{*2}\right]$$

$$\Gamma_5 = \frac{K}{R_2^{*2}}\left(e_1aY^*R_2^{*2} + e_2cZ^*R_1^{*2}\right)\left(\lambda^2Z^*R_1^{*2}R_2^{*2} + e_1e_2ab^2cX^{*2}Z^*\right)$$

Proof. Straightforward computation shows that A_1, A_3 and Δ are positive under the sufficient conditions (15a)-(15e) and hence according to Routh-Hurwitz criterion all the eigenvalues of the $J(E_4)$ have negative real parts. Thus E_4 is locally asymptotically and the proof is complete. ■

IV. BASIN OF ATTRACTION

In this section, we will determine with the help of Lyapunov function the region of all the initial points that approach asymptotically to the equilibrium points (basin of attraction of the equilibrium points) of system (1).

Theorem (3): The predator free equilibrium point $E_1 = (K, 0,0)$ is a globally asymptotically stable in the sub region Ψ_1 of R_+^3 that is defined by

ISSN 2348 – 8034
Impact Factor- 3.155

$$\Psi_1 = \left\{(X,Y,Z) \in R_+^3 : X \geq 0, 0 \leq Y \leq \frac{b\gamma_1}{aK}, 0 \leq Z \leq \frac{b(\gamma_1+\gamma_2)}{cK}\right\}$$

Proof. For any initial value (X,Y,Z) in Ψ_1, define the following positive definite real valued function $V_1 = \left(X - K - K\ln\left(\frac{X}{K}\right)\right) + Y + Z$, that define around E_1. Now by differentiate V_1 with respect to t, we obtain that

$$\frac{dV_1}{dt} < -\frac{r}{K}(X-K)^2 - Y\left(\gamma_1 - \frac{a}{b}KY\right) - Z\left(\gamma_1 + \gamma_2 - \frac{c}{b}KZ\right)$$

Clearly, $\frac{dV_1}{dt}$ is negative definite for any initial point in Ψ_1. Hence E_1 is globally asymptotically stable point in Ψ_1, which is complete the proof. ∎

Theorem (4): The disease free equilibrium point $E_2 = (\bar{X}, \bar{Y}, 0)$ is a globally asymptotically stable in the sub region Ψ_2 of R_+^3 that is defined by

$$\Psi_2 = \left\{(X,Y,Z) \in R_+^3 : X \geq 0, Y \geq 0, 0 \leq Z \leq \frac{(\gamma_1+\gamma_2)-\lambda\bar{Y}}{\frac{c}{b}\bar{X}}\right\}$$

Provided that the following sufficient conditions hold

$$(\bar{R}_1 Y + b\bar{Y} - e_1 bY)^2 < 4e_1 b\bar{Y}^2 Y\bar{X} \quad (16a)$$

$$B < A + C \quad (16b)$$

here $A = -\frac{r}{K}(X-\bar{X})^2$, $B = \frac{a}{R_1\bar{R}_1}\left[\sqrt{\bar{Y}^2 Y}(X-\bar{X}) - \sqrt{e_1 b\bar{X}}(Y-\bar{Y})\right]^2$ and $C = -\left[(\gamma_1+\gamma_2) - \frac{c}{b}\bar{X}Z - \lambda\bar{Y}\right]Z$ with $R_1 = b + XY$ and $\bar{R}_1$ is given in Eq. (12b).

Proof. For any initial value (X,Y,Z) in Ψ_2, define the following positive definite real valued function $V_2 = \left(X - \bar{X} - \bar{X}\ln\left(\frac{X}{\bar{X}}\right)\right) + \left(Y - \bar{Y} - \bar{Y}\ln\left(\frac{Y}{\bar{Y}}\right)\right) + Z$, that define around E_2. Now by differentiate V_2 with respect to t, we obtain that

$$\frac{dV_2}{dt} \leq -\frac{r}{K}(X-\bar{X})^2 - Z\left((\gamma_1+\gamma_2) - \frac{c}{b}\bar{X}Z - \lambda\bar{Y}\right) + \frac{a}{R_1\bar{R}_1}[\bar{Y}^2 Y(X-\bar{X})^2 - (\bar{R}_1 Y + b\bar{Y} - e_1 bY)(X-\bar{X})(Y-\bar{Y}) + e_1 b\bar{X}(Y-\bar{Y})^2]$$

Clearly the second term of the right hand side of the above inequality is negative for any initial point in Ψ_2. While the third term is positive under the condition (16a) that gives that

$$\frac{dV_2}{dt} \leq -\frac{r}{K}(X-\bar{X})^2 - Z\left((\gamma_1+\gamma_2) - \frac{c}{b}\bar{X}Z - \lambda\bar{Y}\right) + \frac{a}{R_1\bar{R}_1}\left[\sqrt{\bar{Y}^2 Y}(X-\bar{X}) - \sqrt{e_1 b\bar{X}}(Y-\bar{Y})\right]^2$$

Finally $\frac{dV_2}{dt}$ is negative definite for any initial point in Ψ_2 under the condition (16b). Hence E_2 is globally asymptotically stable point in Ψ_2, which is complete the proof. ∎

Theorem (5): The susceptible predator free equilibrium point $E_3 = (\hat{X}, 0, \hat{Z})$ is a globally asymptotically stable in the sub region Ψ_3 of R_+^3 that is defined by

$$\Psi_3 = \left\{(X,Y,Z) \in R_+^3 : X \geq 0, 0 \leq Y \leq \frac{\gamma_1 + \lambda\hat{Z}}{\frac{a}{b}\hat{X}}, Z \geq 0\right\}$$

Provided that the following sufficient conditions hold

$$\left(\hat{R}_2 Z + b\hat{Z} - e_2 bZ\right)^2 < 4e_2 b\hat{Z}Z\hat{X} \tag{17a}$$

$$\hat{B} < \hat{A} + \hat{C} \tag{17b}$$

here $\hat{A} = -\frac{r}{K}\left(X - \hat{X}\right)^2$, $\hat{B} = \frac{c}{R_2\hat{R}_2}\left[\sqrt{\hat{Z}^2 Z}\left(X - \hat{X}\right) - \sqrt{e_2 b\hat{X}}\left(Z - \hat{Z}\right)\right]^2$ and $\hat{C} = -\left[\gamma_1 + \lambda\hat{Z} - \frac{a}{b}\hat{X}Y\right]Y$ with $R_2 = b + XZ$ and $\hat{R}_2$ is given in Eq. (13b).

Proof. For any initial value (X, Y, Z) in Ψ_3, define the following positive definite real valued function $V_3 = \left(X - \hat{X} - \hat{X}ln\left(\frac{X}{\hat{X}}\right)\right) + Y + \left(Z - \hat{Z} - \hat{Z}ln\left(\frac{Z}{\hat{Z}}\right)\right)$, that define around E_3. Now by differentiate V_3 with respect to t, we obtain that

$$\frac{dV_3}{dt} \leq -\frac{r}{K}\left(X - \hat{X}\right)^2 - \left[\gamma_1 + \lambda\hat{Z} - \frac{a}{b}\hat{X}Y\right]Y + \frac{c}{R_2\hat{R}_2}\left[\hat{Z}^2 Z\left(X - \hat{X}\right)^2 - \left(\hat{R}_2 Z + b\hat{Z} - e_2 bZ\right)\left(X - \hat{X}\right)\left(Z - \hat{Z}\right) + e_2 b\hat{X}\left(Z - \hat{Z}\right)^2\right]$$

Note that the second term of the right hand side of the above inequality is negative for any initial point in Ψ_3. While the third term is positive under the condition (17a), which leads to:

$$\frac{dV_3}{dt} < -\frac{r}{K}\left(X - \hat{X}\right)^2 - \left[\gamma_1 + \lambda\hat{Z} - \frac{a}{b}\hat{X}Y\right]Y + \frac{c}{R_2\hat{R}_2}\left[\sqrt{\hat{Z}^2 Z}\left(X - \hat{X}\right) - \sqrt{e_2 b\hat{X}}\left(Z - \hat{Z}\right)\right]^2$$

Finally $\frac{dV_3}{dt}$ is negative definite for any initial point in Ψ_3 under the condition (17b). Hence E_3 is globally asymptotically stable point in Ψ_3, which is complete the proof. ■

Theorem (6): The positive equilibrium point $E_4 = (X^*, Y^*, Z^*)$ is a globally asymptotically stable in the sub region Ψ_4 of R^3_+ that is defined by

$$\Psi_4 = \left\{(X, Y, Z) \in R^3_+ : X \geq 0, \frac{b^2}{2X^* R_1^*} < Y, \frac{b^2}{2X^* R_2^*} < Z\right\}$$

Provided that the following sufficient conditions hold

$$\left[\left(\frac{R_1^*}{e_1} - \frac{R_2^*}{e_2}\right)\frac{\lambda}{b}\right]^2 < \frac{acX^{*2}}{R_1 R_2} \tag{18a}$$

$$B^* + C^* + D^* < A^* \tag{18b}$$

Here R_1, R_2, R_1^* and R_2^* as given above. While $A^* = \frac{r}{K}\left(X - \hat{X}\right)^2$, $B^* = \frac{a}{R_1 R_1^*}\left[Y^*\sqrt{Y}(X - X^*) - \sqrt{\frac{X^* R_1^*}{2}}(Y - Y^*)\right]^{\frac{1}{2}}$, $C^* = \frac{c}{R_2 R_2^*}\left[Z^*\sqrt{Z}(X - X^*) - \sqrt{\frac{X^* R_2^*}{2}}(Z - Z^*)\right]^{\frac{1}{2}}$ and $D^* = \left[\sqrt{\frac{aX^*}{2R_1}}(Y - Y^*) - \sqrt{\frac{cX^*}{2R_2}}(Z - Z^*)\right]^{\frac{1}{2}}$.

Proof. For any initial value (X, Y, Z) in Ψ_4, define the following positive definite real valued function that define around E_4

$$V_4 = \left(X - X^* - X^* ln\left(\frac{X}{X^*}\right)\right) + \frac{R_1^*}{e_1 b}\left(Y - Y^* - Y^* ln\left(\frac{Y}{Y^*}\right)\right) + \frac{R_2^*}{e_2 b}\left(Z - Z^* - Z^* ln\left(\frac{Z}{Z^*}\right)\right).$$

Now by differentiate V_4 with respect to t, we obtain that

$$\begin{aligned}\frac{dV_4}{dt} = &-\frac{r}{K}(X - X^*)^2 \\ &+ \frac{a}{R_1 R_1^*}\left[Y^{*2}Y(X - X^*)^2 - bY^*(X - X^*)(Y - Y^*) + \frac{X^* R_1^*}{2}(Y - Y^*)^2\right] \\ &+ \frac{c}{R_2 R_2^*}\left[Z^{*2}Z(X - X^*)^2 - bZ^*(X - X^*)(Z - Z^*) + \frac{X^* R_2^*}{2}(Z - Z^*)^2\right] \\ &+ \left[\frac{aX^*}{2R_1}(Y - Y^*)^2 - \left(\frac{R_1^*}{e_1} - \frac{R_2^*}{e_2}\right)\frac{\lambda}{b}(Y - Y^*)(Z - Z^*) + \frac{cX^*}{2R_2}(Z - Z^*)^2\right]\end{aligned}$$

Therefore for any initial point in the region Ψ_4 and an application of condition (18a) we get that

$$\frac{dV_4}{dt} < -\frac{r}{K}(X - X^*)^2 + \frac{a}{R_1 R_1^*}\left[Y^*\sqrt{Y}(X - X^*) - \sqrt{\frac{X^* R_1^*}{2}}(Y - Y^*)\right]^{\frac{1}{2}} + \frac{c}{R_2 R_2^*}\left[Z^*\sqrt{Z}(X - X^*) - \sqrt{\frac{X^* R_2^*}{2}}(Z - Z^*)\right]^{\frac{1}{2}} + \left[\sqrt{\frac{aX^*}{2R_1}}(Y - Y^*) - \sqrt{\frac{cX^*}{2R_2}}(Z - Z^*)\right]^{\frac{1}{2}}$$

Consequently according to condition (18b) we obtain that $\frac{dV_4}{dt}$ is negative definite for any initial point in Ψ_4. Hence E_4 is globally asymptotically stable point in Ψ_4, which is complete the proof. ■

V. DISCUSSION AND CONCLUSIONS

In this paper, a prey-predator system with vertically transmitted infectious disease in predator population is proposed and analyzed. The existence, uniqueness and boundedness of the solution of the system are discussed. The existence of all possible equilibrium points is studied. The local stability analysis of each of these equilibrium points is investigated. The basin of attraction of all locally stable points is determined. It is observed that, the predator free equilibrium point is always locally asymptotically stable. Indeed it is a globally asymptotically stable in the sub region Ψ_1 that given in theorem (2) above, as shown in the following typical figure, Fig. (1), for the following set of hypothetical biological feasible data.

$$\begin{aligned}&r = 2, K = 400, a = 1, b = 40, c = 0.9, n = 0.1 \\ &e_1 = 0.85, e_2 = 0.4, \gamma_1 = 0.05, \gamma_2 = 0.4\end{aligned} \tag{19}$$

Clearly Fig. (1) shows the global stability of the predator free equilibrium point, due to starting from different sets of initial points. Consequently, system (1) cannot be persist (coexistence of all the species for all the time) in the R_+^3 , and the susceptible predator, infected predator or total predator species should be extinction due to the stability of predator free equilibrium point. On the other hand there is a possibility to get persistence of the system in the specific sub region of R_+^3 under certain conditions such as those given by theorem (6).

Furthermore, its observed that for the data given by Eq. (19) with varying one parameter of the system's parameters at a time don't have qualitative change of the dynamics of system (1) and the trajectory of the system still approaches to predator free equilibrium point. However still there are a possibilities to get a data, which satisfy the existence and stability conditions of the other equilibrium point.

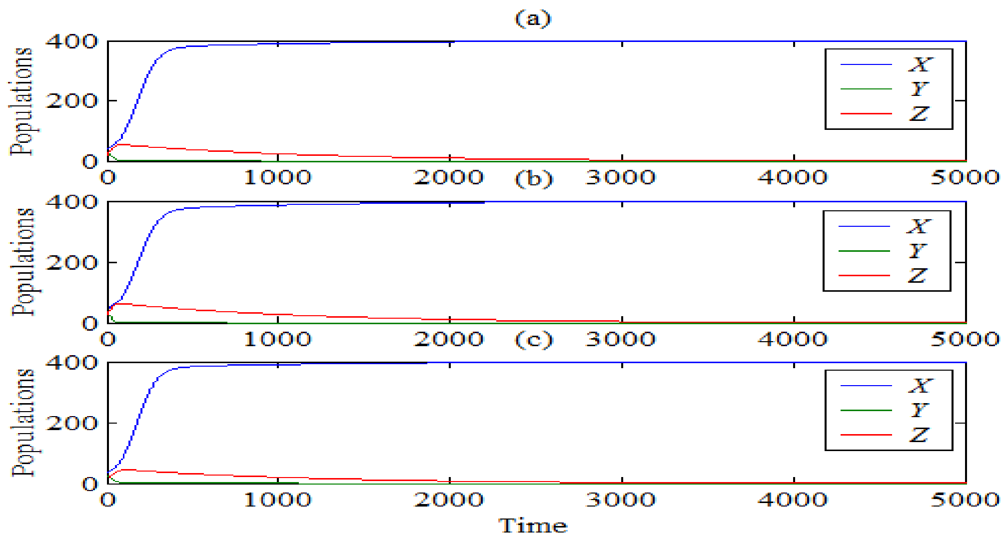

Fig. (1): The trajectory of system (1) approaches asymptotically to the predator free equilibrium point $E_1 = (400, 0, 0)$ for the data given by Eq. (19). (a) The trajectory starting from (40,30.20). (b) The trajectory starting from (45,35.25). (c) The trajectory starting from (35,25.15).

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